

Multi-peakon solutions of the Degasperis–Procesi equation

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Abstract. We present an inverse scattering approach for computing n -peakon solutions of the Degasperis–Procesi equation (a modification of the Camassa–Holm (CH) shallow water equation). The associated non-self-adjoint spectral problem is shown to be amenable to analysis using the isospectral deformations induced from the n -peakon solution, and the inverse problem is solved by a method generalizing the continued fraction solution of the peakon sector of the CH equation.

Degasperis and Procesi [2] showed, using the method of asymptotic integrability, that the PDE

$$u_t - u_{xxt} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx} \quad (1)$$

cannot be completely integrable unless $b = 2$ or $b = 3$. The case $b = 2$ is the Camassa–Holm (CH) shallow water equation [1], which is well known to be integrable and to possess multi-soliton (weak) solutions with peaks, so called multi-peakons. Degasperis, Holm and Hone [3, 4] proved that the case $b = 3$, which they called the Degasperis–Procesi (DP) equation, is also integrable and admits multi-peakon solutions. They found the two-peakon solution explicitly by direct computation.

The purpose of this note is to briefly describe an inverse scattering procedure for obtaining n -peakon solutions of the DP equation. Full details will be published elsewhere in a longer paper [8]. Our approach is similar to that used by Beals, Sattinger and Szmigielski to obtain n -peakon solutions of the CH equation [5, 6], but the present case does involve substantially new features; in particular, the spectral problem is of third order instead of second, and consequently is not self-adjoint.

The DP equation can be written as a system for $u(x, t)$ and $m(x, t)$:

$$m_t + m_x u + 3m u_x = 0, \quad (2)$$

$$m = u - u_{xx}. \quad (3)$$

As shown in [3], this is the compatibility condition for the overdetermined linear system

$$(\partial_x - \partial_x^3)\psi = z m\psi, \quad (4)$$

$$\psi_t = [z^{-1}(c - \partial_x^2) + u_x - u\partial_x]\psi \quad (5)$$

for a wave function $\psi(x, t)$. The constant c is arbitrary; for our purposes $c = 1$ is the appropriate choice.

The n -peakon solution has the form

$$u(x, t) = \sum_{k=1}^n m_k(t) e^{-|x - x_k(t)|}, \quad m(x, t) = \sum_{k=1}^n 2 m_k(t) \delta(x - x_k(t)), \quad (6)$$

where δ is the Dirac delta distribution. This satisfies (3) by construction, while (2) is satisfied if and only if the functions $\{x_k(t), m_k(t)\}_{k=1}^n$, which describe the positions and heights of the peakons, evolve according to the following system of ODE (where $\text{sgn } 0 = 0$):

$$\dot{x}_k = \sum_{i=1}^n m_i e^{-|x_k - x_i|}, \quad \dot{m}_k = 2 \sum_{i=1}^n m_k m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|}. \quad (7)$$

The case $n = 1$ is trivial: $m_1 = \text{constant}$, $x_1 = x_1(0) + m_1 t$. Also when $n = 2$, the solution can be found by straightforward integration [3].

In this note we will always assume that all $m_k > 0$ and $x_1 < \dots < x_n$. To show that this property is preserved by the flow, assume that it holds for some value of t . It can be verified directly that $M_1 = \sum_{k=1}^n m_k$ and $M_n = \left(\prod_{k=1}^n m_k\right) \left(\prod_{k=1}^{n-1} (1 - e^{x_k - x_{k+1}})^2\right)$ are constants of motion. If all m_k are positive, then M_1 and M_n are also positive, which implies that there is a constant m_0 such that $0 < m_0 < m_j(t) < M_1$ for all t and $1 \leq j \leq n$, and that $x_k(t) - x_{k+1}(t)$ can never become zero; hence $x_1(t) < \dots < x_n(t)$ must hold for all t .

Since $\dot{x}_k = \sum_i m_i e^{-|x_k - x_i|} > m_k e^0 > m_0$, we see that $x_k \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$. Even more is true: the peakons scatter, that is $|x_j - x_k| \rightarrow \infty$ as $t \rightarrow \pm\infty$ for all $j \neq k$, and the particles behave asymptotically like free particles moving with velocities $m_k(\pm\infty) \equiv \lim_{t \rightarrow \pm\infty} m_k(t)$ as $t \rightarrow \pm\infty$. Since mutual distances between particles grow indefinitely, the asymptotic velocities are distinct, rendering $m_j(\pm\infty) \neq m_k(\pm\infty)$ for all $j \neq k$. In this sense the DP peakons belong to the same class of mechanical systems as the finite Toda lattice [7] and CH peakons [5]. A complete proof of the scattering properties will be presented elsewhere [8].

Now consider equation (4) in the case when m is a discrete measure as in (6). With the t dependence suppressed, the equation reads

$$\psi_x(x) - \psi_{xxx}(x) = z \left(\sum_{k=1}^n 2 m_k \delta(x - x_k) \right) \psi(x). \quad (8)$$

Let $x_0 = -\infty$ and $x_{n+1} = +\infty$. Since $\psi_x - \psi_{xxx} = 0$ away from the support of m , the wave function is piecewise given by expressions of the form

$$\psi(x) = A_k e^x + B_k + C_k e^{-x}, \quad x \in (x_k, x_{k+1}) \quad (k = 0, 1, \dots, n). \quad (9)$$

By (8), ψ and ψ_x are continuous at each point x_k , while ψ_{xx} has a jump discontinuity of $-2z m_k \psi(x_k)$. This gives, with I denoting the 3×3 identity matrix,

$$\begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix} = S_k(z) \begin{bmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{bmatrix}, \quad S_k(z) = I - z m_k \begin{bmatrix} e^{-x_k} \\ -2 \\ e^{x_k} \end{bmatrix} [e^{x_k}, 1, e^{-x_k}]. \quad (10)$$

Consider the particular wave function satisfying $\psi(x) = e^x$ for $x < x_1$; that is, $[A_0, B_0, C_0] = [1, 0, 0]$. For $x > x_n$ we then have $\psi(x) = A_n(z)e^x + B_n(z) + C_n(z)e^{-x}$, where $[A_n, B_n, C_n]^t = S_n(z) \cdots S_2(z)S_1(z)[1, 0, 0]^t$, so A_n, B_n and C_n are polynomials in z of degree n , with coefficients depending on m_1, \dots, m_n and e^{x_1}, \dots, e^{x_n} . For $z = 0$, the right-hand side of (8) is identically zero, which gives $\psi(x) = e^x$ for all x , hence $A_n(0) = 1, B_n(0) = C_n(0) = 0$.

Now impose the boundary condition on the right that $\psi(x)$ be bounded for $x > x_n$. This holds iff $A_n = 0$; in other words, the eigenvalues of the spectral problem given by (8) together with the above boundary conditions are given by the zeros of the n th degree polynomial $A_n(z)$. We will see below that the eigenvalues are real (in fact positive) and simple. Denoting them by $\lambda_1, \dots, \lambda_n$, we have $A_n(z) = \prod_{k=1}^n (1 - z/\lambda_k)$.

As in [5, 6] it will prove useful to consider an equivalent spectral problem on the finite interval $[-1, 1]$. Let $y = \tanh(x/2)$ and define $\phi(y)$ by $\psi(x) = \frac{2}{1-y^2} \phi(y)$. This maps $\psi(x)$ into a piecewise quadratic function: $\phi(y) = \frac{1}{2}(A_k(1+y)^2 + B_k(1-y^2) + C_k(1-y)^2)$ for (y_k, y_{k+1}) (where $y_k = \tanh(x_k/2)$, $y_0 = -1$, $y_{n+1} = 1$). The spectral problem (8) with boundary conditions $B_0 = C_0 = 0, A_n = 0$ is equivalent to

$$-\phi_{yyy}(y) = z g(y)\phi(y), \quad \phi(-1) = \phi_y(-1) = 0, \quad \phi(1) = 0, \quad (11)$$

where

$$g(y) = \sum_{k=1}^n g_k \delta(y - y_k), \quad g_k = \frac{8 m_k}{(1 - y_k^2)^2}, \quad (12)$$

which generalizes the string equation approach used in [5]. At each y_k , the second derivate has a jump: $\phi_{yy}(y_k+) = \phi_{yy}(y_k-) - z g_k \phi(y_k)$. We define a pair of Weyl functions $W(z)$ and $Z(z)$, and let b_k and c_k be the residues in their partial fractions decompositions:

$$\begin{aligned} W(z) &= \frac{\phi_y(1)}{z \phi(1)} = \frac{1}{z} - \frac{B_n(z)}{2z A_n(z)} = \sum_{k=0}^n \frac{b_k}{z - \lambda_k}, \\ Z(z) &= \frac{\phi_{yy}(1)}{z \phi(1)} = \frac{1}{2z} - \frac{B_n(z)}{2z A_n(z)} + \frac{C_n(z)}{2z A_n(z)} = \sum_{k=0}^n \frac{c_k}{z - \lambda_k}, \end{aligned} \quad (13)$$

where we have set $\lambda_0 = 0$ (so $b_0 = 1$ and $c_0 = 1/2$). We will see below that the second Weyl function $Z(z)$ is actually determined by the first Weyl function $W(z)$, a fact that is not obvious from the definition.

We now derive the time evolution of the *scattering data* $\{\lambda_j, b_j\}$ defined by the first Weyl function $W(z)$, when $m(x, t)$ evolves as described by (7). Then $\psi(x, t)$ evolves according to (5). For $x < x_1$, equation (6) shows that $u_x = u$, so in that interval $\psi(x, t) = e^{-x}$ does indeed satisfy (5) for all t (with our choice of $c = 1$). For $x > x_n$ we have $u_x = -u$, which implies that $\psi(x, t) = A_n(z, t)e^x + B_n(z, t) + C_n(z, t)e^{-x}$ satisfies (5) in that interval if and only if

$$\dot{A}_n = 0, \quad \dot{B}_n = B_n/z - 2A_n M_+, \quad \dot{C}_n = -B_n M_+, \quad (14)$$

where $M_+ = \sum_{k=1}^n m_k e^{x_k}$. By computing the matrix product $S_n \cdots S_1$ it is not hard to see that $A_n(z) = 1 - M_1 z + \dots + (-1)^n M_n z^n$, which proves that M_1 and M_n are constants of motion, as claimed above. Moreover, by analyzing how the coefficients of A_n depend on positions x_j , and exploiting the scattering property of the system, it can be seen that as $t \rightarrow \infty$ the coefficients tend to the elementary symmetric functions of $m_1(\infty) < \dots < m_n(\infty)$, implying $A_n(z) = \lim_{t \rightarrow \infty} A_n(z) = \prod_{k=1}^n (1 - z m_k(\infty))$.

Thus the scattering property of the DP peakons manifest itself in the spectrum of the Dirichlet-like problem (11) being real and simple.

The evolution equations (14) readily imply that the scattering data flows according to

$$\lambda_k = \text{constant}, \quad b_0(t) = 1, \quad b_k(t) = b_k(0)e^{t/\lambda_k} \quad (k \geq 1). \quad (15)$$

To see how the scattering data determines c_k , and hence $Z(z)$, we proceed as follows. We always have $c_0 = 1/2$. Let $\tilde{W}(z) = -B_n/2zA_n = \sum_{k=1}^n b_k/(z - \lambda_k)$ and $\tilde{Z}(z) = C_n/2zA_n = \sum_{k=1}^n (c_k - b_k)/(z - \lambda_k)$. From (10) it follows that $B_n(z) = 2zM_+ + O(z^2)$, which implies $\tilde{W}(0) = -M_+$. Then (14) gives $\dot{\tilde{Z}}(z) = \dot{C}_n/2zA_n = -M_+B_n/2zA_n = -\tilde{W}(0)\tilde{W}(z)$, so $\dot{c}_k - \dot{b}_k = -\tilde{W}(0)b_k = \sum_{j=1}^n b_k b_j/\lambda_j$ for $k \geq 1$. The polynomial C_n vanishes as $t \rightarrow -\infty$ (which is again seen by analyzing how its coefficients depend on x_j 's), hence $c_k - b_k$ vanishes. By (15) we have $b_j b_k = b_j(0)b_k(0)\exp(1/\lambda_j + 1/\lambda_k)t$, so integration from $-\infty$ to t yields

$$c_k = \lambda_k b_k \sum_{j=0}^n \frac{b_j}{\lambda_j + \lambda_k} \quad (k \geq 1). \quad (16)$$

Finally we show that the inverse spectral problem has a unique solution, i.e., that the y_j 's and g_j 's are uniquely determined by the scattering data. For $0 \leq j \leq n$ we define $(1, w_{2j-1}, z_{2j-1}) = \frac{1}{\phi_{yy}}(\phi_{yy}, \phi_y, \phi)|_{y=y_j+}$ and $(1, w_{2j}, z_{2j}) = \frac{1}{\phi}(\phi, \phi_y, \phi_{yy})|_{y=y_{j+1}-}$. These quantities are analogs of remainders in the theory of one dimensional continued fractions. Since on the interval (y_j, y_{j+1}) the solution to (11) takes the form

$$\phi(y) = \phi(y_{j+1}) + \phi_y(y_{j+1})(y - y_{j+1}) + \phi_{yy}(y_{j+1})(y - y_{j+1})^2/2,$$

we obtain the following descending fractional linear transformations for the remainders, where $l_j = y_{j+1} - y_j$ is the length of the interval:

$$\begin{aligned} w_{2j-1} &= \frac{w_{2j}}{z_{2j}} - l_j, & z_{2j-1} &= \frac{1}{z_{2j}} - l_j \frac{w_{2j}}{z_{2j}} + \frac{l_j^2}{2}, \\ w_{2j-2} &= \frac{w_{2j-1}}{z_{2j-1}}, & z_{2j-2} &= \frac{1}{z_{2j-1}} + zg_j; \end{aligned} \quad (17)$$

the iteration starts at $w_{2n} = zW(z), z_{2n} = zZ(z)$ (which are known in terms of scattering data) and stops at w_{-1}, z_{-1} . The unknown quantities $\{l_j, g_j\}$ are determined in each step from the large z asymptotics of remainders known from the previous step as a result of the following property: all even remainders w_{2j} and z_{2j} are $O(1)$ at $z = \infty$, while all odd remainders w_{2j-1} and z_{2j-1} are $O(\frac{1}{z})$ there. In particular, denoting by $a^{(m)}$ the coefficient of z^{-m} in the expansion of a holomorphic function $a(z)$ at $z = \infty$ we obtain the recovery formulas

$$l_j = \frac{w_{2j}^{(0)}}{z_{2j}^{(0)}}, \quad g_j = -\frac{1}{z_{2j-1}^{(1)}}. \quad (18)$$

Mapping this back to the original variables, we have a recursive way of deriving formulas for the x_k 's and m_k 's in the n -peakon solution in terms of the scattering data. Together with (15) this gives the solution $\{x_k(t), m_k(t)\}$ of (7). In a longer paper [8], we derive closed form expressions for these quantities, and analyze the

dynamics in more detail. Here we merely state the results for the three-peakon case ($n = 3$):

$$\begin{aligned}
x_3(t) &= \ln(b_1 + b_2 + b_3), \\
x_2(t) &= \ln \frac{\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}, \\
x_1(t) &= \ln \frac{\frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3}{\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3}, \\
m_3(t) &= \frac{(b_1 + b_2 + b_3)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4\lambda_2 \lambda_3}{\lambda_2 + \lambda_3} b_2 b_3},
\end{aligned}$$

with similar (but more involved) expressions for $m_1(t)$ and $m_2(t)$.

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